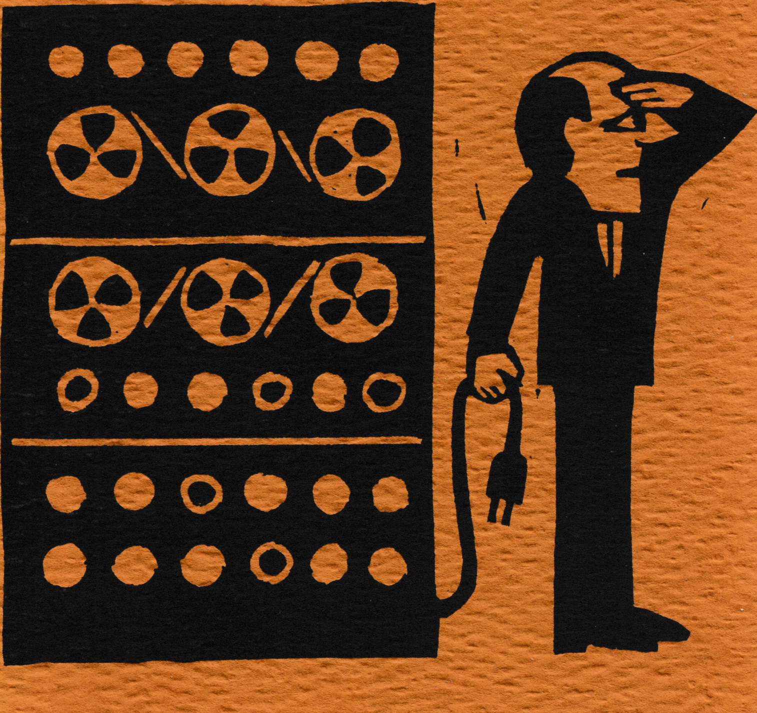
PROBLEMATICAL RECREATIONS^6



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PROBLEMATICAL RECREATIONS^6

Sixth in a Series

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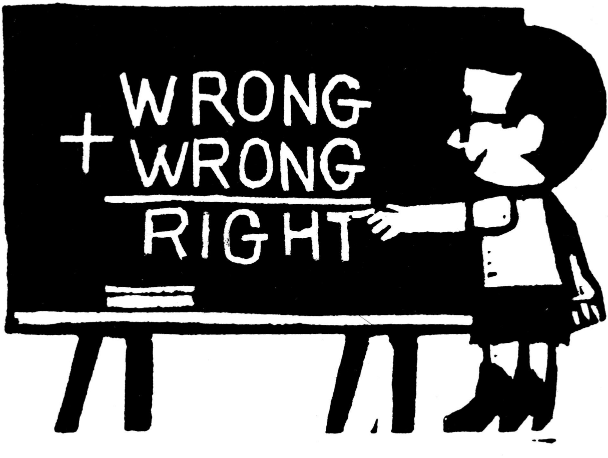
This compilation of mathematical exercises and entertainments represents the original work of dozens of our readers, many of them skilled mathematicians, all of them creative problemists.

You will find that a majority of the answers may be reached by collecting your mental forces, applying your imaginative powers and inventive faculties. When these fail, the reader is advised to turn to the back of the booklet where even the most elusive solution will appear.

⁂

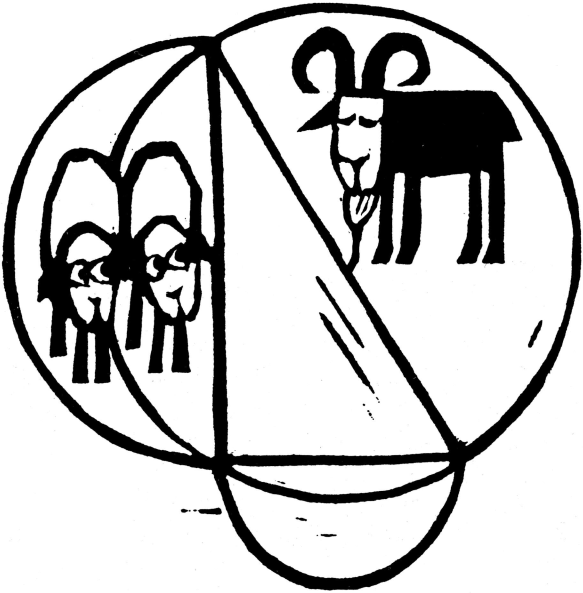
Courtesy of Litton Industries, Beverly Hills, California

1.



Show that two WRONG's *can* make a RIGHT, even with the additional restriction that 0 = zero. *—contributed*

2.



A one-acre field in the shape of a right triangle has a post at the midpoint of each side. A sheep is tethered to each of the side posts and a goat to the post on the hypotenuse. The ropes are just long enough to let each animal reach the two adjacent vertices. What is the total area the two sheep have to themselves, i.e., the area the goat cannot reach? *—contributed*

3.

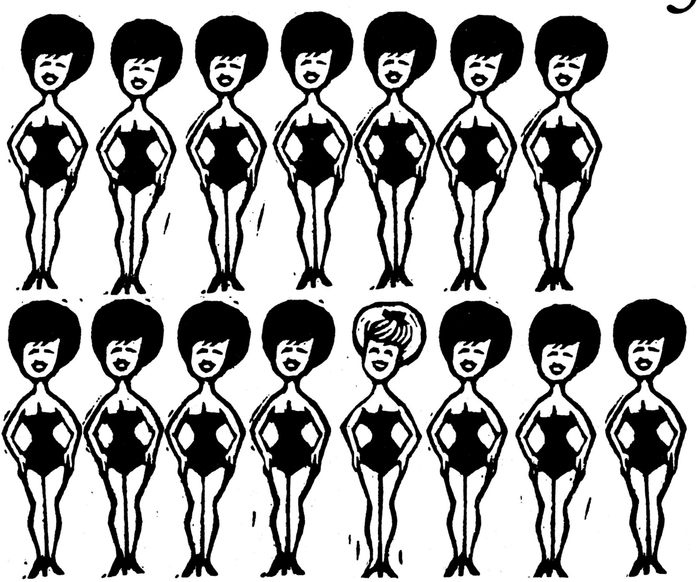
On a certain day, our parking lot contains 999 cars, no two of which have the same 3 digit license number. After 5:00 p.m. what is the probability that the license numbers of the first 4 cars to leave the parking lot are in increasing order of magnitude?

—contributed

4.

Three marksmen simultaneously shoot at and hit a rapidly spinning spherical target. What is the probability that the three points of impact are on the same hemisphere? —*Contributed*

5.



The beauty contest winner had a 36, 23, 34 figure. Although no two contestants had exactly the same measurements, the two runners-up differed less than an inch in each measurement from the winner and the waist of each was ⅔ the hips of the other. If the sum of the three measurements was the same for all three girls, what were the vital statistics of the two runners-up (the tape is accurate only to the quarter-inch). —*Contributed*

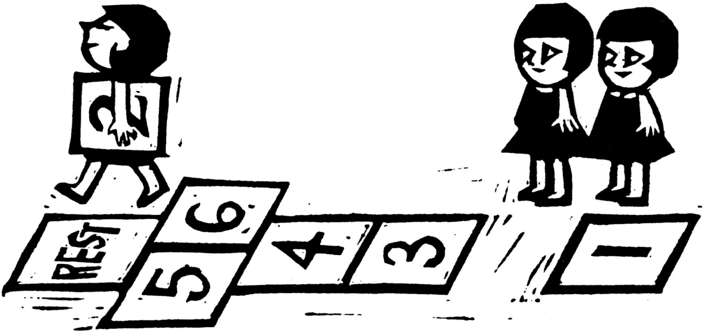
6.

A divided highway goes under a number of bridges, the arch over each lane being in the form of a semi-ellipse with the height equal to the width. A truck is 6 ft. wide and 12 ft. high. What is the lowest bridge under which it can pass? —*Contributed*

7.

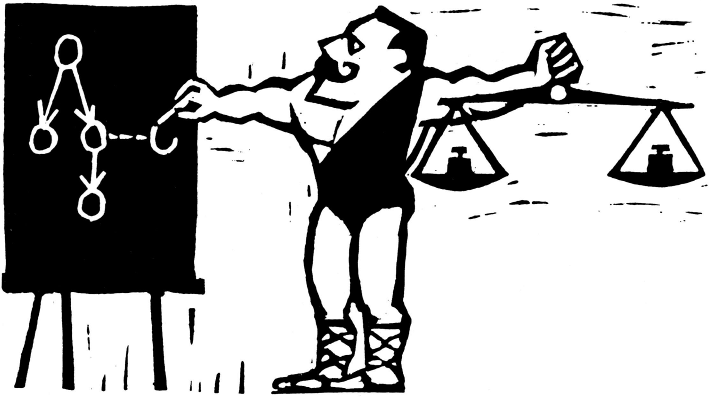
What is the largest amount of money you can have in coins, and still not be able to give change for a dollar? —*Contributed*

8.



In the game "subtract-a-square," a positive integer is written down and two players alternately subtract squares from it with the restriction that the remainder must never be less than zero. The player who leaves zero wins. What square should the first player subtract if the original number is 29? *—Contributed*

9.



There are five weights, no two weighing the same. With a beam balance, arrange the weights in order from heaviest to lightest in seven weighings.

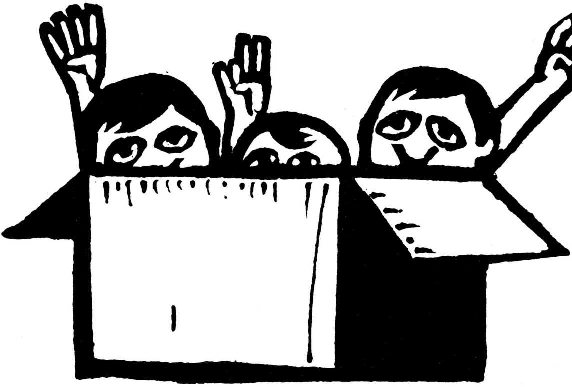
—*Contributed*

10.

The algebra teacher wrote on the blackboard a quadratic equation of the form $x^2 - Ax + B = 0$. In copying this a careless student erroneously transposed the two digits of B as well as the plus and minus signs. However, one of the roots was the same. What was this root?

—Contributed

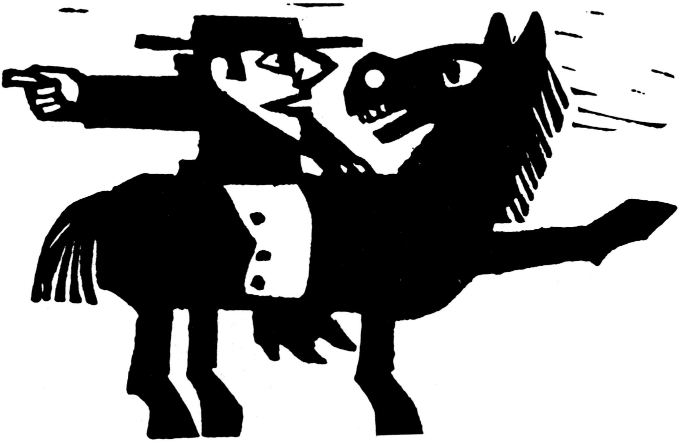
11.



Three times Dick's age plus Tom's age equals twice Harry's age. Double the cube of Harry's age is equal to three times the cube of Dick's age added to the cube of Tom's age. Their respective ages are relatively prime to each other. How old are they?

–Contributed

12.



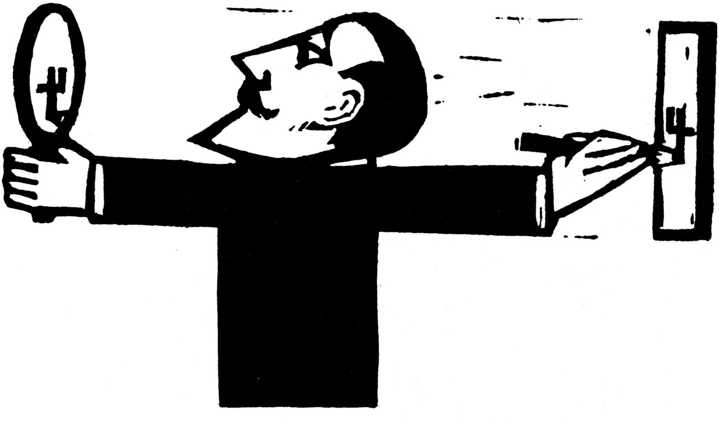
A cowboy is five miles south of a stream which flows due east. He is also 8 miles west and 6 miles north of his cabin. He wishes to water his horse at the stream and return home. What is the shortest distance he can travel and accomplish this?

–Contributed

13.

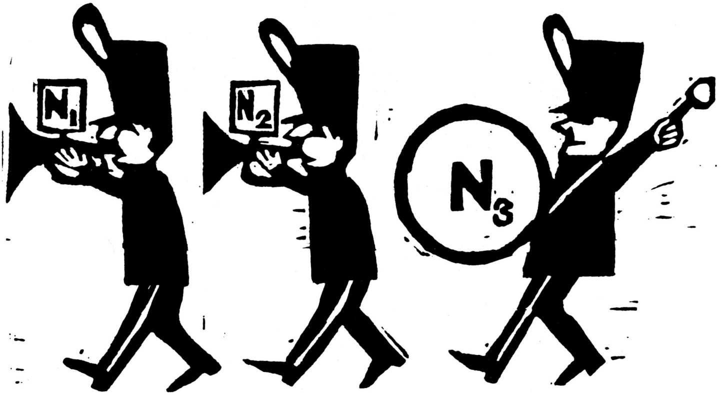
Express $X^{15} + 1$ as the product of sixth and ninth degree polynominals with integral coefficients. *–Contributed*

14.



If a number is added to its reversal and the process repeated with the result, a number will eventually be obtained which reads the same backward and forward. For a certain two-digit number this process must be repeated more than ten times to arrive at a palindromic number. What is this number? *—Contributed*

15.



Find three integers in arithmetic progression whose product is prime.

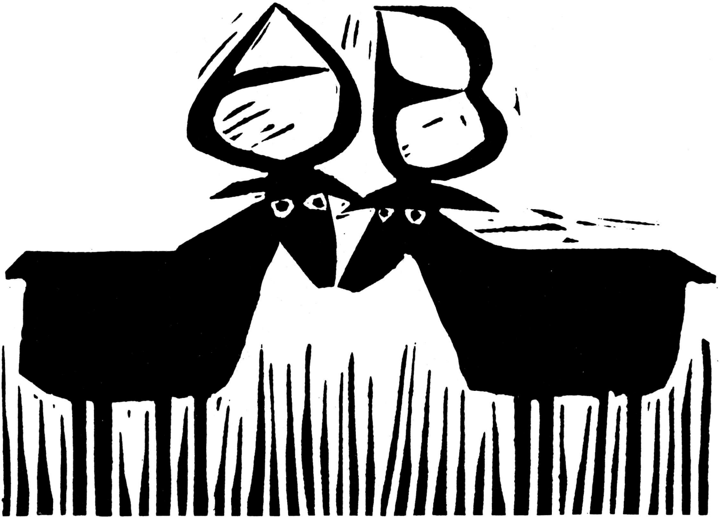
—Contributed

16.

A man is one year older than his wife, and their son is one year older than his wife. The product of the parents' ages is 17 times the product of the ages of their son and daughter-in-law. What are their ages (integers) if their son is not a teen-ager?

—Contributed

17.



Find digits A and B if $A^3 + B^3$ is divisible by the number $10A+B$ and $A^4 + B^4$ is divisible by the number $10B+A$.

—Contributed

18.

On what days of the week can the first day of a century fall? (The first day of the twentieth century was Jan. 1, 1901)

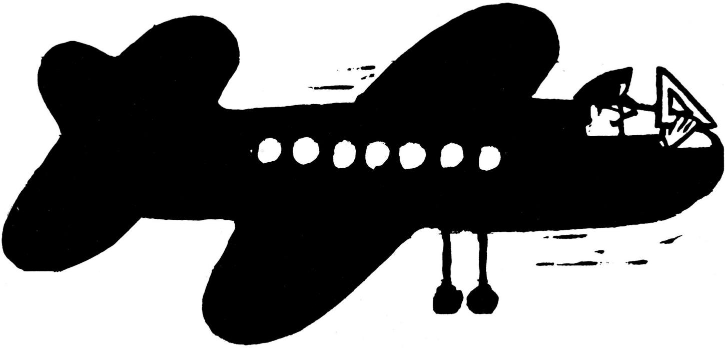
—Contributed

19.

Citizens of Franistan pay as much income tax (percentage-wise) as they make rupees per week. What is the optimal salary in Franistan?

–Contributed

20.



There are nine cities which are served by two competing airlines. One or the other airline (but not both) has a flight between every pair of cities. What is the minimum number of triangular flights (i.e., trips from A to B to C and back to A on the same airline)?

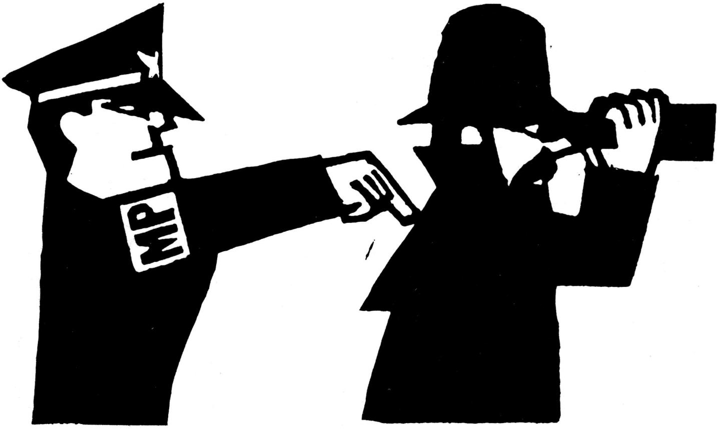
—Contributed

21.

The numbers are divided into three groups as follows: 0,3,6,8,9, ... in the first group, 1,4,7,11,14, ... in the second group and 2,5,10,12,13, ... in the third. In which groups would 15,16 and 17 be placed?

–Contributed

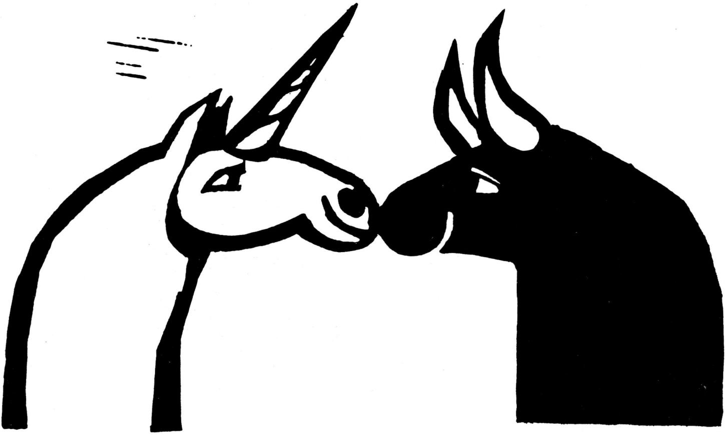
22.



While still at a sizable distance from the Pentagon building, a man first catches sight of it. As the distance from the observer to the Pentagon gets larger, what limiting value does the probability of seeing 2 sides (as opposed to 3) approach?

–Contributed

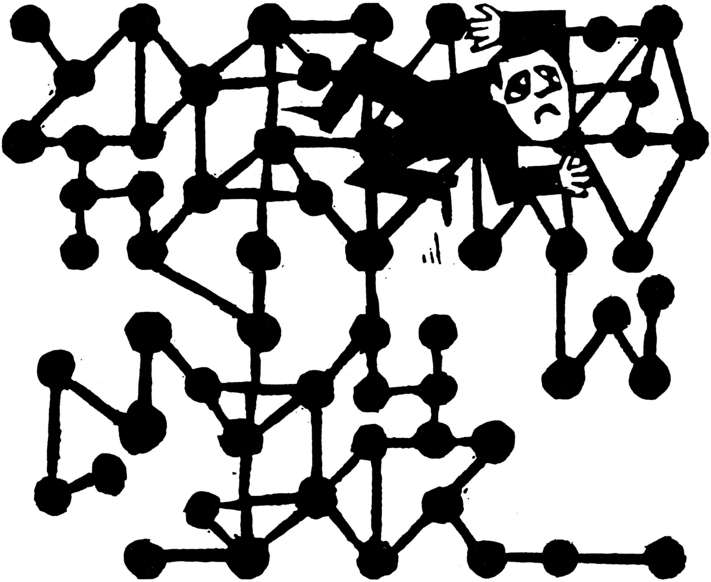
23.



The smaller of two consecutive integers is divisible by 23 and the larger by 29. Find the smallest pair of such numbers with the property that they both contain only the digits one and two.

—Contributed

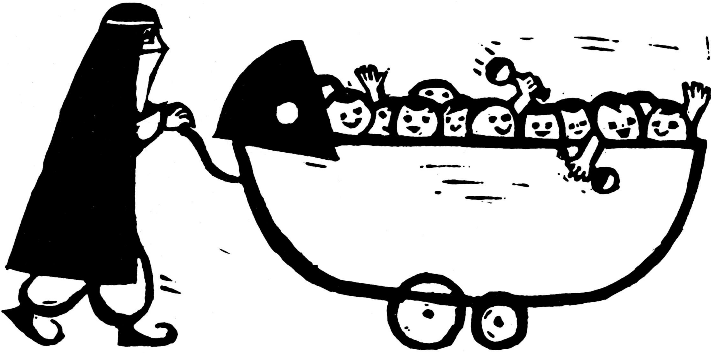
24.



Does the square root of an ATOM extend from A to M? Yes, if the proper numerical values are assigned to the letters. $\sqrt{ATOM} = A + TO + M$. Here ATOM is a four-digit number and TO is a two-digit number.

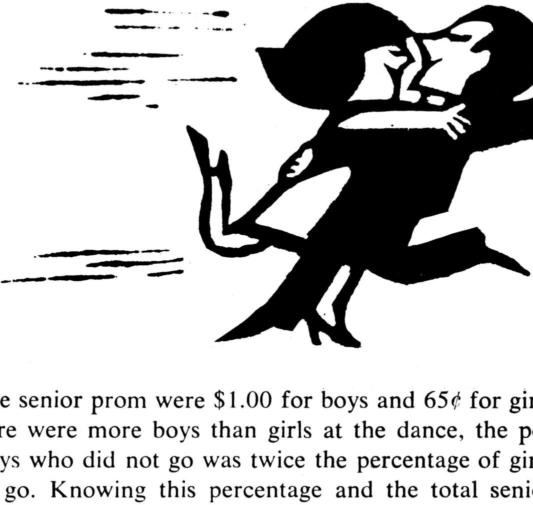
—Contributed

25.



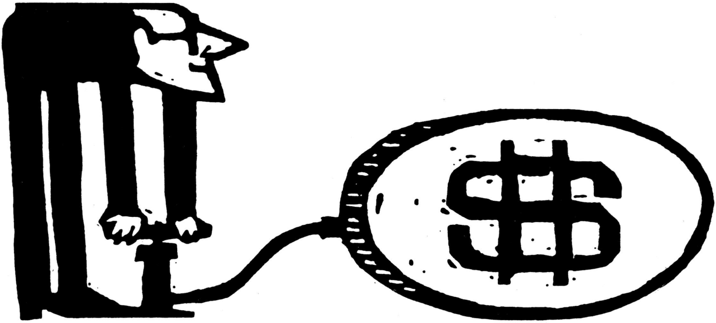
When Maharaja Ram Singh died, he left 3465 gold pieces to be divided equally among his children. Each wife had the same number of children and this number was 8 less than the number of wives per harem, which in turn was 4 more than the number of harems and 4 less than the number of gold pieces each child received. How many children did Ram Singh have? —*Contributed*

26.



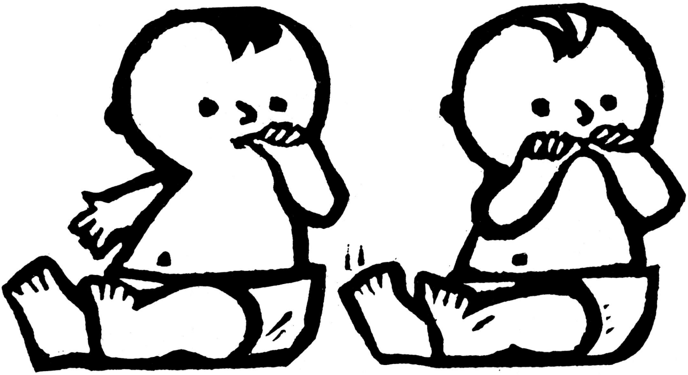
Tickets for the senior prom were \$1.00 for boys and 65¢ for girls. Although there were more boys than girls at the dance, the percentage of boys who did not go was twice the percentage of girls who did not go. Knowing this percentage and the total senior class enrollment, one can deduce the total receipts for the affair. If this enrollment is between 60 and 100, what was the total attendance at the prom? *–Contributed*

27.



Obviously the smaller the compounding period, the greater the interest. How much does one dollar amount to after one year at 100% per annum interest, compounded continuously, i.e., instantaneously? *–Contributed*

28.



It is reckoned that 1/x of the population of a certain country is exclusively left-handed, and that 1/y of the population is ambidextrous, x and y being integers. The number of exclusively left-handed persons exceeds that of the ambidextrous ones by the smallest possible number consistent with the fact that both groups together constitute 1/91 of the population. What fraction of the population is exclusively left-handed and what fraction is ambidextrous?

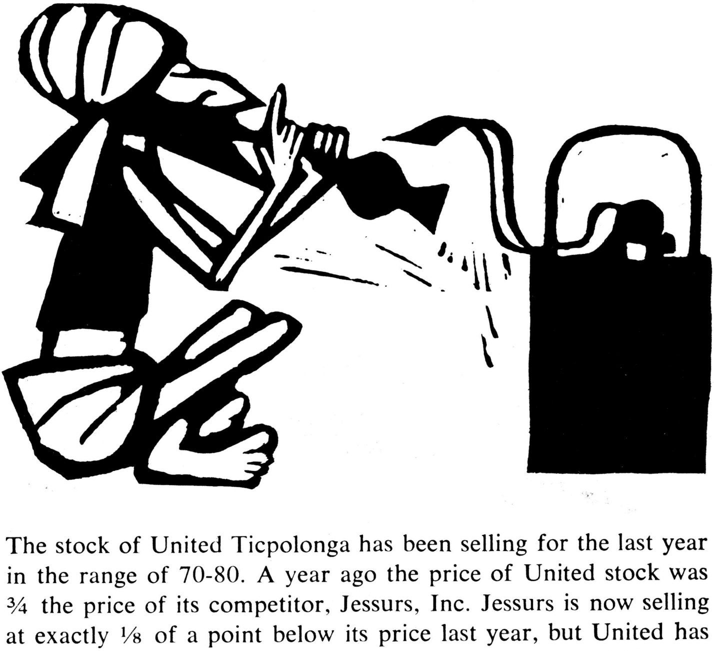
–Contributed

29.

Three rectangles of integer sides have identical areas. The first rectangle is 278 feet longer than wide. The second rectangle is 96 feet longer than wide. The third rectangle is 542 feet longer than wide. Find the area and dimensions of the rectangles.

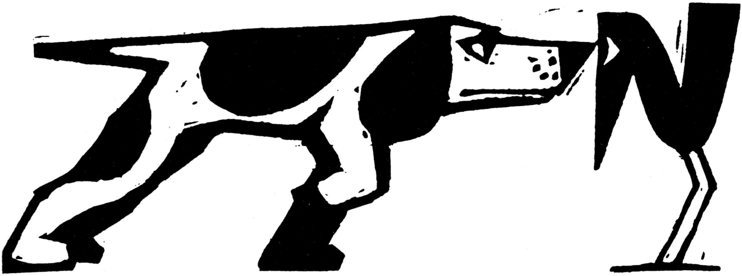
—Contributed

30.



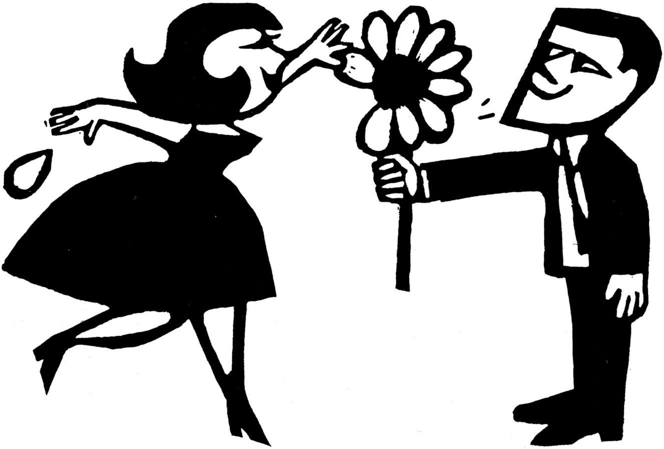
The stock of United Ticpolonga has been selling for the last year in the range of 70-80. A year ago the price of United stock was ¾ the price of its competitor, Jessurs, Inc. Jessurs is now selling at exactly ⅛ of a point below its price last year, but United has declined so far it is only ⅔ the price of Jessurs now. What are the two currently selling at? *—Contributed*

31.



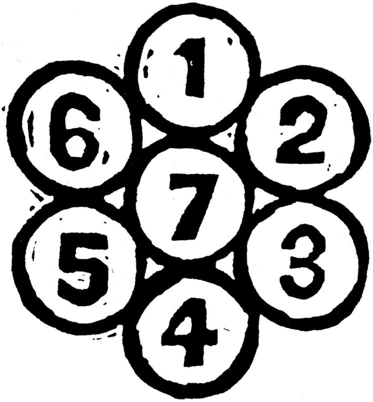
Find N if N^3 and N^4 together contain the ten digits 0 – 9 once and only once. *–Contributed*

32.



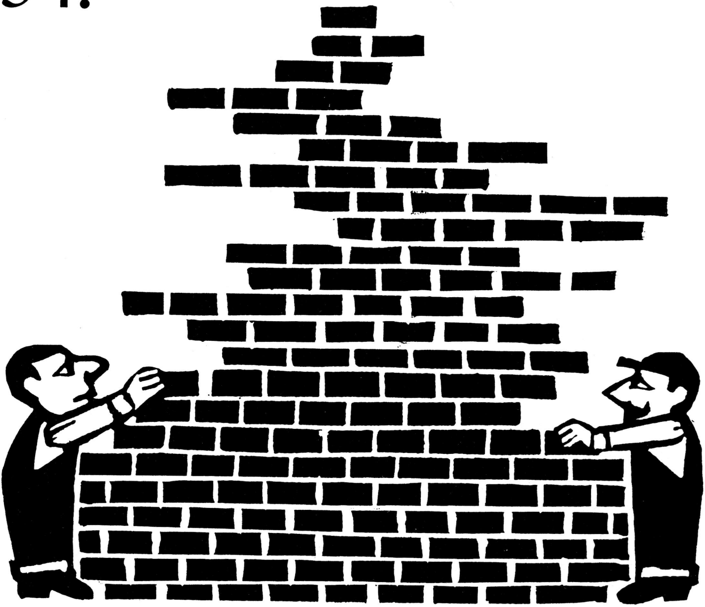
Strephon and Phyllis decide to test their love with a daisy. They agree to pluck petals alternately, taking either one petal or two adjacent petals. There are 13 petals altogether. He picks one saying, "She loves me." She picks two adjacent petals, leaving two groups of 8 and 2, saying, "He loves me not." How should Strephon continue? —*Contributed*

33.



There are 120 7-digit numbers which can be formed by starting from any number in the above diagram and proceeding to any neighboring number, using each number once and only once. Of these, how many are divisible by 11? *—Contributed*

34.



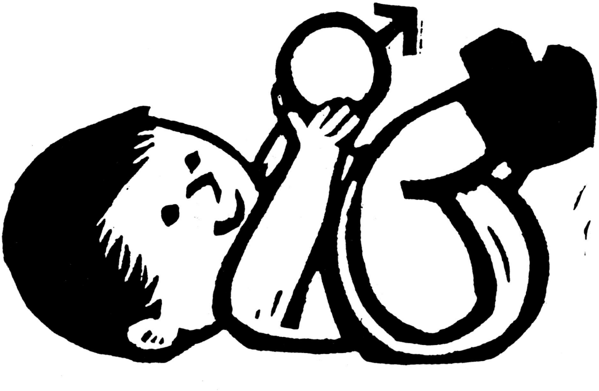
A contractor estimated that one of his two bricklayers would take 9 hours to build a certain wall and the other 10 hours. However, he knew from experience that when they worked together, 10 fewer bricks got laid per hour. Since he was in a hurry, he put both men on the job and found it took exactly 5 hours to build the wall. How many bricks did it contain? *—Contributed*

35.

Every proper fraction can be expressed as the sum of a number of distinct aliquot fractions, i.e., fractions with unit numerators. What is the "smallest" fraction (sum of numerator and denominator a minimum) which requires four such fractions for its expression?

—*Contributed*

36.

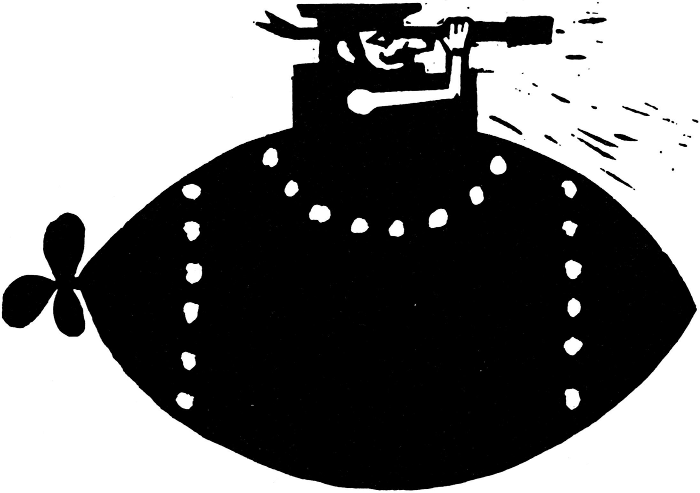


A hospital nursery contains only two baby boys; the girls have not yet been counted. At 2:00 p.m. a new baby is added to the nursery. A baby is then selected at random to be the first to have its footprint taken. It turns out to be a boy. What is the probability that the last addition to the nursery was a girl? *—Contributed*

37.

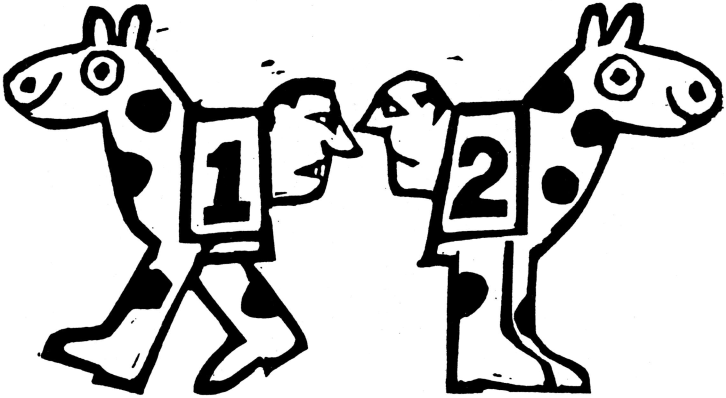
A pirate buried his treasure on an island, a conspicuous landmark of which were three palm trees, each one 100 feet from the other two. Two of these trees were in a N-S line. The directions for finding the treasure read: "Proceed from southernmost tree 15 feet due north, then 26 feet due west." Is the treasure buried within the triangle formed by the trees? —*Contributed*

38.



Assume that a single depth charge has a probability of 1/2 of sinking a submarine, 1/4 of damage and 1/4 of missing. Assume also that two damaging explosions sink the sub. What is the probability that 4 depth charges will sink the sub? —*Contributed*

39.



No two-digit number times its reversal equals any other two-digit number times its reversal. But two numbers come close – in fact the products differ by 4. What are the numbers? – *Contributed*

40.

What is the most likely distribution of the suits in a hand at Bridge? (It is not 4-3-3-3.) *—Contributed*

ANSWERS

1. $\begin{array}{r}37081\\ \underline{37081}\\ 74162\end{array}$ and $\begin{array}{r}37091\\ \underline{37091}\\ 74182\end{array}$

2. The two sheep have exactly one acre to themselves.

3. There are 4! or 24 possible permutations of 4 cars. Only one of these is in increasing rank of license magnitude. Thus there is one chance in 24. The number of cars in the lot (999) is irrelevant.

4. The probability is one, since any three points on the surface of a sphere are always located on some hemisphere.

5. 36, 22½, 34½ and 36¼, 23, 33¾.

6. We have $\frac{x^2}{a^2}+\frac{y^2}{4a^2}=1$ as the equation of the ellipse, where 2 a is the height of the arch. From the data, $x=3$, $y=12$, so that $a=3\sqrt{5}$. The truck will therefore be able to go under an arch $6\sqrt{5}$ or approximately 13 ft. 5 in high.

7. $1.19.

8. To insure a win the first player must subtract 9. He then counters his opponents' plays of 1, 4, 9, or 16 with 9, 16, 9, or 4 respectively.

9. Weigh A against B, C against D, then match the two heavier ones. Without loss of generality assume A heavier than B and C, with C heavier than D. This can be accomplished in 3 weighings. Now weigh E against C, and suppose E heavier than C, in which case weigh E against A (5th weighing). If A is heavier than E, then match B against C, followed by a match between B and E or B and D depending on whether B is heavier or lighter than C. If E is heavier than A, match B against C and against D if necessary, which completes the ordering in 7 or less weighings. The case E lighter than C can be handled in a like manner.

10. The equation must have been either $x^2-11x+24=0$ (which was copied as $x^2+11x-42=0$) or $x^2-22x+57=0$ (which was copied as $x^2+22x-75=0$). In either case the common root is 3.

11. It will be found that where the ages are relatively prime only one set of numbers is possible: Dick is one, Tom is five, and Harry is four years old.

12. The distance will be a minimum if the two legs of his trip make equal angles with the stream. This will be the case if he reaches the stream at a point 2½ miles downstream from the point directly opposite. The total distance will then be $8\sqrt{5}$ or approximately 17.9 miles.

13. $X^{15}+1=(X^6-2X^5+3X^4-3X^3+3X^2-2X+1)\,(X^9+2X^8+X^7-X^6-2X^5-2X^4-X^3+X^2+2X+1)$.

14. Any number with two different digits, having the sum of its digits less than 10 will produce a palindromic number in one operation. We need consider therefore only numbers the sum of whose digits is 10 or greater. If the sum of the digits is 10, 11, 12, 13, 14, 15 or 16, the number of operations required will be 2, 1, 2, 2, 3, 4, 6 respectively. Only when the sum of the digits is 17 are more than 10 operations required. But the only number having 17 as the sum of its digits is 89 (or is reversal, 98).

15. $-3, -1, +1$.

16. The parents are 85 and 84; the son and daughter-in-law are 21 and 20.

17. $A = 3$, $B = 7$.

18. If D is the number of the day in year Y, $X = D + Y + \left[\frac{Y-1}{4}\right] - \left[\frac{Y-1}{100}\right] + \left[\frac{Y-1}{400}\right]$ (mod 7) is the day of the week D falls on, (Sun. = 1, Mon. = 2, etc.). If $D = 1$, $Y = 100\,N + 1$, this becomes $X = 124\,N + 2 + \left[\frac{N}{4}\right] \equiv 5N + 2 + \left[\frac{N}{4}\right]$ (mod 7). This congruence has only the solutions 0,2,3, and 5. Therefore the first day of a century can fall only on Mon., Tues., Thurs., or Sat. The 21st century will start on a Monday.

19. On a salary of $50 \pm X$ rupees, a Franistanean pays $\left(\frac{50 \pm X}{100}\right) \cdot (50 \pm X)$ rupees and is left with $25 - \frac{X^2}{100}$ rupees. Since $X = 0$ maximizes the take home pay, 50 rupees is the optimal salary.

20. If the number of cities is of the form $4\,n + 1$, the minimum number of triangular flights is $\binom{2\,n}{3} + \binom{2\,n+1}{3} - n$. In the problem $n = 2$. Therefore $\binom{4}{3} + \binom{5}{3} - 2$ or $4 + 10 - 2 + 12$ is the number of triangular flights. This minimum number can be obtained from many different arrangements. One matrix corresponding to a complete nonagon might look like this:

	1	2	3	4	5	6	7	8	9
1		A	B	A	A	A	B	B	B
2			B	B	A	B	A	B	A
3				A	A	B	B	A	A
4					B	B	A	A	B
5						A	B	B	B
6							B	A	A
7								A	A
8									B

Here the cities are numbered from 1 to 9 and the two airlines are A and B, the table showing the airline serving any two cities. There are exactly 12 triangles which are either all A or all B.

21. The first group consists of numbers written only with curves, the second group consists of numbers written only with straight lines and the third group consists of numbers written with both straight lines and curves. Therefore 15 and 16 would be in the third group and 17 in the second.

22. Assume the observer is at an infinite distance from the Pentagon and that another man, diametrically opposite, is also observing the building. If the first man can see two sides, the second man can see three sides, and vice versa. Therefore, in the limiting case, the chance of seeing two sides must be the same as the chance of seeing three sides and since only two or three sides can be seen at once, each of these probabilities must be one-half.

23. Let $N = 23a$, $N + 1 = 29b$. Then $29b = 23a + 1$, from which a is of the form $29k + 5$ and b is of the form $23k + 4$. Obviously N ends in 1, so that 23a ends in 1 and a ends in 7. Therefore k must end in 8. We find $k = 18$, $a = 527$, $b = 418$, so that the two numbers are 12121 and 12122.

24. $\sqrt{1296} = 1 + 29 + 6$ or $\sqrt{6724} = 6 + 72 + 4$.

25. The Maharaja had 231 children.

26. Let N = total number in senior class, of which B are boys and let p = percentage of girls who did not go to the dance. We have then: Total receipts $= B\ (1 - 2\ p) + .65\ (N - B)\ (1 - p)$ or $.35\ B - 1.35\ B\ p + .65\ N - .65\ N\ p$. If one can deduce the total receipts from a knowledge of p and N, this expression must be independent of B; therefore $p = 7/27$. Since the total enrollment is between 60 and 100, $B = 54$ and the number of girls = 27, so that there were 26 boys and 20 girls or 46 altogether at the dance.

27. $A = \lim_{t \to 0} (1 + t)^{1/t}$ where t is the compounding period in years. Thus A = \$e = \$2.71^{+}.

28. We must solve $1/91 = 1/x + 1/y$. If $91 = abc$, then $x = ac\ (a + b)$ and $y = bc\ (a + b)$. This leads to four sets of solutions, of which the set we want is 1/140 and 1/260, which are the fractions of exclusively left-handed and ambidextrous people respectively.

29. Area is 1,466,640 square feet. Rectangular dimensions are: 1080 by 1358; 1164 by 1260; and 970 by 1512 feet.

30. If we let U_0 and U_1 be the price of United stock a year ago and today, we have $3/2\ U_1 = 4/3\ U_0 - \frac{1}{8}$. U_0 must be equal to or less than 80, U_1 must be equal to or greater than 70. Since stock market prices are only quoted to eighths, $U_0 = 79\frac{1}{8}$, $U_1 = 70\frac{1}{4}$ is the only solution. The present prices are, therefore: United 70¼, Jessurs 105⅜.

31. If $N \geqq 22$, N^3 will contain at least 5 digits, N^4 at least 6, so that they will contain at least 11 digits together. Likewise if $N \leqq 17$, N^3 and N^4 will contain at most 9 digits together. Hence $N = 18$, 19, 20 or 21. Trying each in turn, we find only $18^3 = 5832$, $18^4 = 104976$. Therefore $N = 18$.

32. Strephon may pick one of the end petals from the group of 8 making two groups of 7 and 2. If then Phyllis leaves 7, he can make this 3,3 and if she leaves 2,2,4, he can make this 2,2,1, 1, in both cases winning since he can duplicate her later moves. If she leaves anything else he can convert it into 1,2,3, and win. Strephon can also win by leaving groups of 8 and 1, or of 6,1, and 2, or of 4,3, and 2. The analysis for these three winning moves is not difficult and is left to the reader.

33. For a number to be divisible by 11, the sum of the odd placed digits must be congruent (mod 11) to the sum of the even placed digits. Since the sum of all seven digits is 28, we seek partitions of 14 into three and four parts, using the numbers 1 to 7 once and only once. There are exactly ten such numbers subject to the given conditions: 1237456, 1237654, 1675432, 2345761, 3217456, 3217654, 4567123, 4567321, 6547123, and 6547321.

34. Let $N =$ Number of bricks in wall, $\frac{N}{9} =$ Number of bricks first bricklayer lays per hour, $\frac{N}{10} =$ Number of bricks second bricklayer lays per hour, $\frac{N}{9} + \frac{N}{10} - 10 =$ Number of bricks laid per hour when they work together, and finally $\frac{N}{N/9 + N/10 - 10} = 5$, from which $N = 900$.

35. $\frac{8}{11} = \frac{1}{2} + \frac{1}{5} + \frac{1}{37} + \frac{1}{4070}$. It is readily proved that $\frac{8}{11}$ cannot be expressed with fewer than 4 aliquot fractions, while every "smaller" fraction can be so expressed.

36. Let GA and BF denote the respective events: girl baby arrives at 2:00 p.m., boy baby selected first for footprinting. Then P (GA) • P (BF, given GA) = P (GA and BF) = P (BF) • P (GA, given GF) or $\frac{1}{2} \cdot \frac{2}{g+3} = \frac{1}{2}\left(\frac{2}{g+3} + \frac{3}{g+3}\right)$ • P (GA, given BF), where g is the original number of girl babies. Thus P (GA, given BF) = 2/5 and is independent of the number of girl babies.

37. The two parts of the route to the treasure form a right triangle 15 feet on one side and 26 feet on the other. If the third tree lies to the east of the other two, the treasure will certainly lie outside the 'tree-angle'. Otherwise, since the angle opposite the 26 side is arctan $\frac{26}{15}$, hence greater than 60°, whereas the trees form angles of 60° with each other, the treasure will lie outside the tree-angle also. Therefore, regardless of the orientation of the trees, the treasure will lie outside the triangle they form.

38. The sub escapes if 4 depth charges miss $\left(\text{probabiity } \frac{1}{4^4} = \frac{1}{256}\right)$ or if 3 miss and one damages $\left(\text{probability} = \frac{4}{4^4}\right)$. Hence the sub escapes with probability $\frac{5}{256}$ and is sunk with probability $\frac{251}{256}$.

39. If the numbers are a, b and c, d, we have $(a + 10\,b)\,(10\,a + b) = (c + 10\,d)\,(10\,c + d) - 4$. $10\,[(a + b)^2 - (c + d)^2] = 81\,(c\,d - a\,b) - 4$. $cd - ab$ must be of the form $10k + 4$ and by test, 4 is seen to be the only possible value. Then $(a + b)^2 - (c + d)^2$ must be 32. The only squares whose difference is 32 are 81 and 49 or 36 and 4. The second of these leads to no solution for $c\,d - a\,b = 4$. The pair 81 and 49 leads to $a = 1$, $b = 8$, $c = 3$, $d = 4$. The numbers are, therefore, 18 and 34. $18 \times 81 = 1458$, $34 \times 43 = 1462$.

40. There are $\binom{13}{4}$ or 715 ways of choosing 4 cards in a suit, $\binom{13}{3}$ or 286 ways of choosing 3 cards in a suit and $\binom{13}{2}$ or 78 ways of choosing 2 cards in a suit. There are, in addition, $\left(\frac{4!}{2!\;1!\;1!}\right)$ ways of assigning the suits in a 4-4-3-2 hand, and $\left(\frac{4!}{3!\;1!}\right)$ ways of assigning the suits in a 4-3-3-3 hand. The total number of hands having these distributions are, therefore, $12(715)^2 \cdot (286) \cdot (78)$ and $4 \cdot (715) \cdot (286)^3$ respectively, or 136,852,887,600 and 66,905,856,160, more than two to one in favor of the 4-4-3-2 distribution.

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